Recursion operator for stationary Nizhnik–Veselov–Novikov equation

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Abstract

We present a new general construction of recursion operator from zero curvature representation. Using it, we find a recursion operator for the stationary Nizhnik–Veselov–Novikov equation and present a few low order symmetries generated with the help of this operator.

In the present letter we suggest a new method of construction of recursion operator using the zero curvature representation. Unlike the majority of the hitherto known methods, see, e.g., [1]–[4] and references therein, ours is immediately applicabile to both evolutionary and non-evolutionary systems, and gives not only the recursion operator, but also its inverse, leading to the 'negative' part of the hierarchy of the system in question. We apply the method to the stationary Nizhnik–Veselov–Novikov (NVN) equation for which no recursion operator has been found so far.

Let $F = 0$ be a system of PDEs in two independent variables x, y for the unknown n-component vector function $\boldsymbol{u} = (u^1, \ldots, u^n)^T$, where the superscript 'T' denotes matrix transposition. Let this system have a zero curvature representation $D_y(A) - D_x(B) + [A, B] = 0$, where A and B take values in a (matrix) Lie algebra $\mathfrak{g}$ and depend on λ, x, y and $\boldsymbol{u}$ and its derivatives. Here D_x, D_y are the operators of total x- and y-derivatives, see, e.g., Ch 2 of [5], and [6]. Note that A and B may involve an essential (spectral) parameter λ.

Consider a (possibly vector or matrix) function P of x, y, $\boldsymbol{u}$ and its derivatives. Then the directional derivative of P along an n-component vector $\boldsymbol{U} = (U^1, \ldots, U^n)^T$ is given by $P'[\boldsymbol{U}] = \sum_{\alpha=1}^{n} \sum_{i,j=0}^{\infty} (\partial P/\partial u^\alpha_{ij}) D_x^i D_y^j (U^\alpha)$, where $u^\alpha_{00} \equiv u^\alpha$, $u^\alpha_{ij} = \partial^{i+j} u^\alpha / \partial x^i \partial y^j$, cf, e.g., [7]. In [6] $P'[\boldsymbol{U}]$ is called a linearization and denoted $\ell_P \boldsymbol{U}$.

Let $\boldsymbol{U}$ be a symmetry of the system $F = 0$, that is, let $\boldsymbol{U}$ satisfy $F'[\boldsymbol{U}] = 0$ on the solution manifold of $F = 0$ [5, 6]. Consider a $\mathfrak{g}$-valued solution S of the system

$$D_x(S) - [A, S] = \tilde{A} \equiv A'[\boldsymbol{U}], \quad D_y(S) - [B, S] = \tilde{B} \equiv B'[\boldsymbol{U}]. \qquad (1)$$

Assume that we have found n linear combinations $\tilde{U}^\alpha = \sum_{i,j} a^{\alpha,ij} S_{ij}$ of entries S_{ij} of S, $\alpha = 1, \ldots, n$, with the property that $\tilde{\boldsymbol{U}} = (\tilde{U}^1, \ldots, \tilde{U}^n)^T$ is another symmetry of $F = 0$. Then

the linear operator $\mathfrak{R}_0$ defined by $\tilde{\boldsymbol{U}} = \mathfrak{R}_0(\boldsymbol{U})$ is a recursion operator, in Guthrie's [8] sense, for the equation $F = 0$. The coefficients $a^{\alpha,ij}$ may depend on λ, x, y and $\boldsymbol{u}$ and its derivatives.

However, testing the above scheme on a number of known examples like KdV or Harry Dym equation shows that $\mathfrak{R}_0$ generates the nonlocal symmetries that belong to the 'negative' part of the hierarchy of $F = 0$. Then we should, if possible, invert $\mathfrak{R}_0$ in order to obtain a 'conventional' recursion operator $\mathfrak{R}$, which will generate the 'positive', local part of the hierarchy in question. The inversion is an algorithmic process described in [8]. Note [9] that if the coefficients of the recursion operator are local, then so are the coefficients of its inverse.

Let us now apply this procedure to the stationary NVN equation

$$u_{yyy} = u_{xxx} - 3(vu)_x + 3(wu)_y, \quad w_x = u_y, \quad v_y = u_x. \tag{2}$$

recently studied by Ferapontov [10], see also Rogers and Schief [11], in connection with isothermally asymptotic surfaces in projective differential geometry.

The stationary NVN equation is a reduction of the NVN equation [12, 13]

$$u_t = u_{xxx} - u_{yyy} - 3(vu)_x + 3(wu)_y, \quad w_x = u_y, \quad v_y = u_x, \tag{3}$$

obtained upon assuming that u, v, w are independent of t. The latter is well known to be integrable via the inverse scattering transform, as it has the Lax pair

$$\psi_{xy} = u\psi, \quad \psi_t = \psi_{xxx} - \psi_{yyy} - 3v\psi_x + 3w\psi_y. \tag{4}$$

The NVN equation (3) is the first member of the hierarchy describing the deformations preserving the zero energy level of two-dimensional Schrödinger operator [13]. It also naturally arises in the theory of surfaces, see [11] and references therein, and its modified version appears in the string theory [14, 15].

Upon setting [11] $\psi = \tilde{\psi}\exp(\lambda t)$, where λ is a constant, the Lax pair (4) can be transformed into a zero-curvature representation for (2) of the form $D_y(A) - D_x(B) + [A, B] = 0$. This representation involves an essential parameter λ, and the matrices A and B belong to the semisimple Lie algebra sl_6 of traceless 6×6 matrices. They read

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ u & 0 & 0 & 0 & 0 & 0 \\ A_{41} & \lambda & u_y & 0 & 0 & -u \\ 0 & 3v & 0 & -1 & 0 & 0 \\ u_y & 0 & u & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ B_{41} & u_x & 0 & 0 & -u & 0 \\ u_x & u & 0 & 0 & 0 & 0 \\ \lambda & 0 & 3w & -1 & 0 & 0 \end{pmatrix}, \tag{5}$$

matrices with zero trace, and where $A_{41} = -u_{yy} + 3wu$, $B_{41} = -u_{xx} + 3vu$.

Let $\boldsymbol{U} = (U, V, W)^T$ be a symmetry of (2), i.e. let U, V, W satisfy

$$\begin{aligned} D_y^3 U &= D_x^3 U + 3\left[wD_yU + u(D_yW - D_xV) - u_xV + u_yW + (w_y - v_x)U - vD_xU\right], \\ D_xW &= D_yU, \quad D_yV = D_xU. \end{aligned} \tag{6}$$

Consider a traceless 6×6 matrix S that solves (1), where A, B are given by (5) and

$$\tilde{A}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ u & 0 & 0 & 0 & 0 & 0\\ \tilde{A}_{41} & 0 & D_yU & 0 & 0 & -U\\ 0 & 3V & 0 & 0 & 0 & 0\\ D_yU & 0 & U & 0 & 0 & 0 \end{pmatrix},\quad \tilde{B}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ U & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \tilde{B}_{41} & D_xU & 0 & 0 & -U & 0\\ D_xU & U & 0 & 0 & 0 & 0\\ 0 & 0 & 3W & 0 & 0 & 0 \end{pmatrix}$$

are the directional derivatives of the matrices A and B (5) along the vector $(U,V,W)^T$, $\tilde{A}_{41}=-D_y^2(U)+3wU+3uW$, $\tilde{B}_{41}=-D_x^2(U)+3vU+3uV$.

The next step is to find linear combinations $\tilde{U},\tilde{V},\tilde{W}$ of entries of S that solve (6). A straightforward but tiresome computation shows that for A and B (5) these are $\tilde{U}=-S_{35}+S_{26}$, $\tilde{V}=S_{54}+S_{12}$, $\tilde{W}=-S_{13}-S_{64}$, i.e. if $\boldsymbol{U}=(U,V,W)^T$ is a symmetry of (2), then so is $\tilde{\boldsymbol{U}}=(\tilde{U},\tilde{V},\tilde{W})^T$. Hence, the linear operator $\mathfrak{R}_0$ mapping $\boldsymbol{U}$ to $\tilde{\boldsymbol{U}}$ is a recursion operator for (2).

However, the application of $\mathfrak{R}_0$ to the simplest symmetries of (2), e.g. to the zero symmetry, yields nonlocal symmetries of (2), so we should invert $\mathfrak{R}_0$ in order to obtain a recursion operator $\tilde{\mathfrak{R}}=\mathfrak{R}_0^{-1}$ generating hierarchies of local symmetries for (2).

It turns out that our $\mathfrak{R}_0$ is invertible only for $\lambda\neq 0$. Inverting $\mathfrak{R}_0$ involves solving a system of algebraic and differential equations for the components of $\tilde{\mathfrak{R}}$, which is a fairly tiresome but algorithmic process. For the sake of simplicity we set all the integration constants to zero. Then $\mathfrak{R}=\lambda\tilde{\mathfrak{R}}-\frac{1}{2}\lambda^2\,\mathrm{id}$, where id is the identity operator, is independent of λ and provides a conventional recursion operator for (2).

The action of $\mathfrak{R}$ on a symmetry $\boldsymbol{U}=(U,V,W)^T$ of (2) is given by $\mathfrak{R}(\boldsymbol{U})=\mathfrak{L}(\boldsymbol{U})+\mathfrak{M}\vec{Z}$. Here $\vec{Z}=(Z_1,Z_2,Z_3,Z_4,Z_5)^T$ is a general solution of the system

$$\begin{aligned}
&D_xZ_1=U,\quad D_yZ_1=W,\quad D_xZ_2=V,\quad D_yZ_2=U,\\
&D_xZ_3=D_y^2U-3(Wu+Uw),\quad D_yZ_3=D_x^2U-3(Vu+Uv),\\
&D_xZ_4=-\tfrac{1}{3}vD_y^2U+\tfrac{1}{3}u_xD_yU+(-\tfrac{1}{3}u_{xy}+u^2+vw)U+(-\tfrac{1}{3}u_{yy}+\tfrac{1}{3}v_{xx}\\
&\quad+2uw-v^2)V+uvW+uu_xZ_1+(wu_x+uu_y-vv_x)Z_2,\\
&D_yZ_4=-\tfrac{1}{3}vD_x^2U+\tfrac{1}{3}v_xD_xU+(-\tfrac{1}{3}u_{yy}+2uw)U+uvV+uu_yZ_1+u^2W\\
&\quad+(-vu_x+wu_y+uw_y)Z_2,\\
&D_xZ_5=-\tfrac{1}{3}wD_y^2U+\tfrac{1}{3}w_yD_yU+(-\tfrac{1}{3}u_{xx}+2uv)U+u^2V+uwW\\
&\quad+(vu_x-wu_y+uv_x)Z_1+uu_xZ_2,\\
&D_yZ_5=-\tfrac{1}{3}wD_x^2U+\tfrac{1}{3}u_yD_xU+(-\tfrac{1}{3}u_{xy}+u^2+vw)U+uwV\\
&\quad+(-\tfrac{1}{3}u_{xx}+\tfrac{1}{3}w_{yy}+2uv-w^2)W+(uu_x+vu_y-ww_y)Z_1+uu_yZ_2.
\end{aligned}\tag{7}$$

Note that this system is compatible if and only if $\boldsymbol{U}$ solves (6).

The operators $\mathfrak{L}$ and $\mathfrak{M}$ are of the form

$$\mathfrak{L}=\begin{pmatrix}\mathfrak{L}_{11} & \mathfrak{L}_{12} & \mathfrak{L}_{13}\\ \mathfrak{L}_{21} & \mathfrak{L}_{22} & \mathfrak{L}_{23}\\ \mathfrak{L}_{31} & \mathfrak{L}_{32} & \mathfrak{L}_{33}\end{pmatrix},\quad \mathfrak{M}=\begin{pmatrix}\mathfrak{M}_{11} & \mathfrak{M}_{12} & \mathfrak{M}_{13} & \mathfrak{M}_{14} & \mathfrak{M}_{15}\\ \mathfrak{M}_{21} & \mathfrak{M}_{22} & \mathfrak{M}_{23} & \mathfrak{M}_{24} & \mathfrak{M}_{25}\\ \mathfrak{M}_{31} & \mathfrak{M}_{32} & \mathfrak{M}_{33} & \mathfrak{M}_{34} & \mathfrak{M}_{35}\end{pmatrix},$$

$$\begin{aligned}
\mathfrak{L}_{11} &= D_x^6 - 6vD_x^4 - \tfrac{25}{9}uD_x^2D_y^2 - 15v_xD_x^3 - \tfrac{2}{9}u_yD_x^2D_y - \tfrac{29}{9}u_xD_xD_y^2 + (-\tfrac{5}{3}u_{yy} - 18v_{xx} \\
&\quad + \tfrac{40}{3}uw + 9v^2)D_x^2 + 9u^2D_xD_y + (-\tfrac{5}{3}u_{xx} + \tfrac{13}{3}uv)D_y^2 + (-3u_{xyy} - 12v_{xxx} \\
&\quad + \tfrac{56}{3}wu_x + 26uu_y + 27vv_x)D_x + (26uu_x + \tfrac{2}{3}vu_y)D_y - 3u_{xxyy} - 3v_{xxxx} \\
&\quad + 14wu_{xx} + 20uu_{xy} + 5vu_{yy} + 9vv_{xx} + \tfrac{77}{3}u_xu_y + 9v_x^2 - 4u^3 - 28uvw, \\
\mathfrak{L}_{12} &= -\tfrac{28}{9}uD_x^4 - \tfrac{106}{9}u_xD_x^3 + (-\tfrac{55}{3}u_{xx} + \tfrac{32}{3}uv)D_x^2 + (-\tfrac{44}{3}u_{xxx} + \tfrac{74}{3}vu_x + 18uv_x)D_x \\
&\quad - 6u_{xxxx} + 19vu_{xx} + 4uu_{yy} + 10uv_{xx} + \tfrac{79}{3}u_xv_x + \tfrac{2}{3}u_y^2 - 12u^2w - 4uv^2, \\
\mathfrak{L}_{13} &= -\tfrac{1}{9}uD_y^4 + \tfrac{2}{9}u_yD_y^3 + (-\tfrac{1}{3}u_{yy} + \tfrac{5}{3}uw)D_y^2 + (\tfrac{1}{3}u_{xxx} - vu_x - \tfrac{4}{3}wu_y - uv_x + uw_y)D_y \\
&\quad + 13uu_{xx} + wu_{yy} + uw_{yy} + \tfrac{29}{3}u_x^2 - \tfrac{2}{3}u_yw_y - 12u^2v - 4uw^2, \\
\mathfrak{L}_{21} &= \tfrac{28}{27}D_x^4D_y^2 - \tfrac{28}{9}wD_x^4 - 6uD_x^3D_y - \tfrac{32}{9}vD_x^2D_y^2 - \tfrac{86}{9}u_yD_x^3 - \tfrac{134}{9}u_xD_x^2D_y - \tfrac{58}{9}v_xD_xD_y^2 \\
&\quad + (-\tfrac{148}{9}u_{xy} + \tfrac{28}{3}u^2 + \tfrac{32}{3}vw)D_x^2 + (-\tfrac{154}{9}u_{xx} + \tfrac{46}{3}uv)D_xD_y + (-\tfrac{16}{9}u_{yy} \\
&\quad - \tfrac{28}{9}v_{xx} + \tfrac{16}{3}uw + \tfrac{4}{3}v^2)D_y^2 + (-14u_{xxy} + \tfrac{89}{3}uu_x + \tfrac{50}{3}vu_y + \tfrac{58}{3}wv_x)D_x \\
&\quad + (-\tfrac{86}{9}u_{xxx} + \tfrac{58}{3}vu_x + \tfrac{59}{3}uv_x)D_y - 6u_{xxxy} + 20uu_{xx} + \tfrac{40}{3}vu_{xy} + \tfrac{16}{3}wu_{yy} \\
&\quad + \tfrac{28}{3}wv_{xx} + \tfrac{47}{3}u_x^2 + 19u_yv_x - 12u^2v - 16uw^2 - 4v^2w, \\
\mathfrak{L}_{22} &= -\tfrac{1}{27}D_x^6 + \tfrac{2}{3}vD_x^4 + v_xD_x^3 + (-\tfrac{28}{9}u_{yy} + \tfrac{11}{9}v_{xx} + \tfrac{28}{3}uw - 3v^2)D_x^2 + (-\tfrac{56}{9}u_{xyy} \\
&\quad + \tfrac{10}{9}v_{xxx} + \tfrac{56}{3}wu_x + \tfrac{56}{3}uu_y - 7vv_x)D_x - \tfrac{40}{9}u_{xxyy} + \tfrac{5}{9}v_{xxxx} + \tfrac{40}{3}wu_{xx} \\
&\quad + \tfrac{52}{3}uu_{xy} + \tfrac{16}{3}vu_{yy} - \tfrac{16}{3}vv_{xx} + \tfrac{70}{3}u_xu_y - 2v_x^2 - 4u^3 - 16uvw + 4v^3, \\
\mathfrak{L}_{23} &= -\tfrac{1}{9}u_xD_y^3 + (\tfrac{1}{3}u_{xy} - \tfrac{1}{3}u^2)D_y^2 + (-\tfrac{2}{3}u_{xyy} + \tfrac{5}{3}wu_x + 2uu_y)D_y - \tfrac{19}{9}u_{xxxx} + \tfrac{23}{3}vu_{xx} \\
&\quad - wu_{xy} + \tfrac{13}{3}uu_{yy} + \tfrac{19}{3}uv_{xx} + \tfrac{40}{3}u_xv_x + \tfrac{4}{3}u_xw_y - 12u^2w - 4uv^2,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{L}_{31} &= \tfrac{28}{27}D_x^5D_y - 6uD_x^4 - \tfrac{56}{9}vD_x^3D_y - \tfrac{4}{9}wD_x^2D_y^2 - \tfrac{142}{9}u_xD_x^3 + (-\tfrac{28}{3}v_x - \tfrac{2}{9}w_y)D_x^2D_y \\
&\quad - \tfrac{22}{9}u_yD_xD_y^2 + (-\tfrac{184}{9}u_{xx} + \tfrac{70}{3}uv + \tfrac{4}{3}w^2)D_x^2 + (-\tfrac{14}{9}u_{yy} - \tfrac{28}{3}v_{xx} + \tfrac{20}{3}uw \\
&\quad + \tfrac{28}{3}v^2)D_xD_y + (-\tfrac{8}{9}u_{xy} + \tfrac{2}{3}u^2 + \tfrac{4}{3}vw)D_y^2 + (-\tfrac{142}{9}u_{xxx} + \tfrac{116}{3}vu_x + \tfrac{28}{3}wu_y \\
&\quad + \tfrac{82}{3}uv_x + uw_y)D_x + (-\tfrac{14}{9}u_{xyy} - \tfrac{28}{9}v_{xxx} + \tfrac{20}{3}wu_x + \tfrac{37}{3}uu_y + \tfrac{28}{3}vv_x \\
&\quad + \tfrac{2}{3}vw_y)D_y - 6u_{xxxx} + \tfrac{70}{3}vu_{xx} + \tfrac{14}{3}wu_{xy} + \tfrac{8}{3}uu_{yy} + \tfrac{52}{3}uv_{xx} + \tfrac{82}{3}u_xv_x \\
&\quad + \tfrac{1}{3}u_xw_y + 7u_y^2 - 12u^2w - 16uv^2 - 4vw^2, \\
\mathfrak{L}_{32} &= -\tfrac{29}{9}u_yD_x^3 + (-9u_{xy} + \tfrac{25}{3}u^2)D_x^2 + (-10u_{xxy} + \tfrac{88}{3}uu_x + 11vu_y)D_x - \tfrac{47}{9}u_{xxxy} \\
&\quad + \tfrac{58}{3}uu_{xx} + \tfrac{44}{3}vu_{xy} + \tfrac{4}{3}wu_{yy} + \tfrac{49}{3}u_x^2 + \tfrac{23}{3}u_yv_x + \tfrac{2}{3}u_yw_y - 12u^2v - 4uw^2, \\
\mathfrak{L}_{33} &= -\tfrac{1}{27}D_y^6 + \tfrac{2}{3}wD_y^4 + w_yD_y^3 + (\tfrac{11}{9}w_{yy} - 3w^2)D_y^2 + (\tfrac{10}{9}w_{yyy} - 7ww_y)D_y - \tfrac{4}{3}u_{xxyy} \\
&\quad + \tfrac{5}{9}w_{yyyy} + \tfrac{16}{3}wu_{xx} + \tfrac{26}{3}uu_{xy} + 4vu_{yy} - \tfrac{16}{3}ww_{yy} + \tfrac{40}{3}u_xu_y - 2w_y^2 \\
&\quad - 4u^3 - 16uvw + 4w^3; \\
\mathfrak{M}_{11} &= -u_{xxxyy} + 2wu_{xxx} + \tfrac{14}{3}uu_{xxy} + 3vu_{xyy} + \tfrac{1}{3}uw_{yyy} + \tfrac{23}{3}u_yu_{xx} + 9u_xu_{xy} + 3v_xu_{yy} \\
&\quad + w_yu_{yy} - \tfrac{2}{3}u_yw_{yy} - 4u^2u_x - 6vwu_x - 8uvu_y - 6uwv_x - 4uww_y, \\
\mathfrak{M}_{12} &= -u_{xxxxx} + 5vu_{xxx} + \tfrac{5}{3}uu_{xyy} + \tfrac{10}{3}uv_{xxx} + 10v_xu_{xx} + \tfrac{5}{3}u_xu_{yy} + \tfrac{25}{3}u_xv_{xx} - 8uwu_x \\
&\quad - 6v^2u_x - 4u^2u_y - 10uvv_x, \\
\mathfrak{M}_{13} &= -\tfrac{2}{3}u_{xxx} + 2vu_x + 2uv_x, \quad \mathfrak{M}_{14} = -2u_x, \quad \mathfrak{M}_{15} = -2u_y, \\
\mathfrak{M}_{21} &= -u_{xxxxy} + 5uu_{xxx} + 3vu_{xxy} + 2wu_{xyy} + \tfrac{32}{3}u_xu_{xx} + 6v_xu_{xy} - w_yu_{xy} + u_yu_{yy} \\
&\quad + 3u_yv_{xx} + \tfrac{1}{3}u_xw_{yy} - 9uvu_x - 6w^2u_x - 9uwu_y - 5u^2v_x + u^2w_y, \\
\mathfrak{M}_{22} &= -\tfrac{10}{9}u_{xxxyy} + \tfrac{1}{9}v_{xxxxx} + \tfrac{10}{3}wu_{xxx} + 5uu_{xxy} + \tfrac{10}{3}vu_{xyy} - \tfrac{5}{3}vv_{xxx} + \tfrac{25}{3}u_yu_{xx} + 10u_xu_{xy} \\
&\quad + \tfrac{10}{3}v_xu_{yy} - \tfrac{5}{3}v_xv_{xx} - 10vwu_x - 4u^2u_x - 10uvu_y + 4v^2v_x - 8uwv_x, \\
\mathfrak{M}_{23} &= -\tfrac{2}{3}u_{xyy} + 2wu_x + 2uu_y, \quad \mathfrak{M}_{24} = -2v_x, \quad \mathfrak{M}_{25} = -2u_x, \\
\mathfrak{M}_{31} &= -\tfrac{10}{9}u_{xxxxx} + \tfrac{1}{9}w_{yyyyy} + \tfrac{20}{3}vu_{xxx} + \tfrac{5}{3}uu_{xyy} + \tfrac{10}{3}uv_{xxx} - \tfrac{5}{3}ww_{yyy} + 10v_xu_{xx} + \tfrac{5}{3}u_xu_{yy} \\
&\quad + 10u_xv_{xx} - \tfrac{5}{3}w_yw_{yy} - 10v^2u_x - 10uwu_x - 4u^2u_y - 10uvv_x + 4w^2w_y + 2uvw_y, \\
\mathfrak{M}_{32} &= -u_{xxxxy} + 5uu_{xxx} + 5vu_{xxy} + 10u_xu_{xx} + 5v_xu_{xy} + \tfrac{5}{3}u_yu_{yy} + \tfrac{10}{3}u_yv_{xx} \\
&\quad - 15uvu_x - 6v^2u_y - 3uwu_y - 5u^2v_x + u^2w_y, \\
\mathfrak{M}_{33} &= -\tfrac{2}{3}u_{xxy} + 2uu_x + 2vu_y, \quad \mathfrak{M}_{34} = -2u_y, \quad \mathfrak{M}_{35} = -2w_y.
\end{aligned}$$

Note that the above formula $\mathfrak{R}(\boldsymbol{U}) = \mathfrak{L}(\boldsymbol{U}) + \mathfrak{M}\vec{Z}$ defines a recursion operator in the sense of Guthrie [8], and the system (7) defines a covering [6] over (6). Formally, we could express Z_i from (7) as $Z_1 = D_x^{-1}U, Z_2 = D_x^{-1}V$ etc., and thus write $\mathfrak{R}$ as an integro-differential operator, as it became traditional in the literature, see, e.g., [3, 5, 7]. However, if we drop the y-part of (7), we encounter certain difficulties in constructing new symmetries, cf, e.g., [8, 17, 18].

As integrating (7) involves arbitrary constants, we have $\mathfrak{R}(0) = c_1\boldsymbol{S}_1 + c_2\boldsymbol{S}_2 + \cdots + c_5\boldsymbol{S}_5$, where c_i are constants, and $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_5$ are symmetries of (2) of the following form:

$$\boldsymbol{S}_1 = \begin{pmatrix} \mathfrak{M}_{11} + \frac{1}{2}u^2\mathfrak{M}_{14} + (uv - \frac{1}{2}w^2)\mathfrak{M}_{15} \\ \mathfrak{M}_{21} + \frac{1}{2}u^2\mathfrak{M}_{24} + (uv - \frac{1}{2}w^2)\mathfrak{M}_{25} \\ \mathfrak{M}_{31} + \frac{1}{2}u^2\mathfrak{M}_{34} + (uv - \frac{1}{2}w^2)\mathfrak{M}_{35} \end{pmatrix}, \quad \boldsymbol{S}_2 = \begin{pmatrix} \mathfrak{M}_{12} + (uw - \frac{1}{2}v^2)\mathfrak{M}_{14} + \frac{1}{2}u^2\mathfrak{M}_{15} \\ \mathfrak{M}_{22} + (uw - \frac{1}{2}v^2)\mathfrak{M}_{24} + \frac{1}{2}u^2\mathfrak{M}_{25} \\ \mathfrak{M}_{32} + (uw - \frac{1}{2}v^2)\mathfrak{M}_{34} + \frac{1}{2}u^2\mathfrak{M}_{35} \end{pmatrix},$$
$$\boldsymbol{S}_3 = \begin{pmatrix} -u_{xxx} + 3(vu_x + uv_x) \\ -u_{xyy} + 3(wu_x + uu_y) \\ -u_{xxy} + 3(vu_y + uu_x) \end{pmatrix}, \quad \boldsymbol{S}_4 = \boldsymbol{u}_x \equiv \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix}, \quad \boldsymbol{S}_5 = \boldsymbol{u}_y \equiv \begin{pmatrix} u_y \\ v_y \\ w_y \end{pmatrix}.$$

The repeated application of $\mathfrak{R}$ to $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_5$ produces five hierarchies of symmetries of the stationary NVN equation (2), which can be visualized as follows (numbers in the top line denote the orders of symmetries):

$$\begin{array}{ccccccccc}
 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & \\
 & \boldsymbol{S}_5 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_5 & \longrightarrow & \longrightarrow & \mathfrak{R}^2\boldsymbol{S}_5 & \cdots \\
 & \boldsymbol{S}_4 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_4 & \longrightarrow & \longrightarrow & \mathfrak{R}^2\boldsymbol{S}_4 & \cdots \\
0 & \longrightarrow & \boldsymbol{S}_3 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_3 & \longrightarrow & \longrightarrow & \cdots \\
 & \longrightarrow & \longrightarrow & \boldsymbol{S}_2 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_2 & \longrightarrow & \cdots \\
 & \longrightarrow & \longrightarrow & \boldsymbol{S}_1 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_1 & \longrightarrow & \cdots
\end{array}$$

We conjecture that all these symmetries are local and commute, as it is the case for the symmetries of orders $1, 3, 5, \ldots, 11$.

Note that (2) has a scaling symmetry $\boldsymbol{S} = x\boldsymbol{u}_x + y\boldsymbol{u}_y + 2\boldsymbol{u}$. The application of $\mathfrak{R}$ to $\boldsymbol{S}$ yields a nonlocal symmetry of seventh order, which we conjecture to be a master symmetry for (2), meaning that commuting $\mathfrak{R}(\boldsymbol{S})$ with any symmetry belonging to one of the five hierarchies, described above, yields (up to a constant multiplier) the next member of the same hierarchy. The repeated application of $\mathfrak{R}$ to $\boldsymbol{S}$ yields an infinite hierarchy of nonlocal symmetries for (2).

We believe that $\mathfrak{R}$ is hereditary in the sense of [16], but we have not yet checked this because of the huge amount of computations involved.

As a final remark, let us mention the nonstandard structure of nonlocal terms of $\mathfrak{R}$ in (7): they involve the derivatives of components of the symmetry, what is quite unusual, cf, e.g., [19] for another example of this kind and [20] for a comprehensive list of known today integrable systems in (1+1) dimensions and their recursion operators.

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